Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Differential Equations

Instructor : Yogeshwaran D.

Date : March 5th, 2015.

Answer for 50 marks. **Time Limit : 3 hours.**

Give necessary justifications and explanations for all your arguments.
If you are citing results from the class/book, mention it clearly.
Convergence of series needs to be justified.

1. Let an object be dropped from an height of 300m. The object is falling under gravity with air-resistance proportional to velocity exerted upon the object. Please formulate the differential equation as an initial value problem. Let T be the time required for the object to fall to ground. Find the equation governing T i.e., a function $f : [0, \infty) \to \mathbb{R}$ such that $f(T) = 0$. (**6**)

2. Consider the following two initial value problems (IVPs) for some $x_0, y_0 \in \mathbb{R}^2$: (**5**)

   (a) $y^{'} = y|y|$; $y(x_0) = y_0$.

   (b) $y^{'} = y^{\frac{1}{3}} + x$; $y(x_0) = y_0$.

   (a) For what points (x_0, y_0) does there exist a $h > 0$ such that the above two IVP's have a solution on $[x_0 - h, x_0 + h]$?

   (b) For what points (x_0, y_0) does there exist a $h > 0$ such that the above two IVP's have a unique solution on $[x_0 - h, x_0 + h]$?

   Note that h is allowed to differ for the two IVPs.

3. Show that if $\frac{(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})}{Ny - Mx}$ is a funnction $g(z)$ of the product $z = xy$, then

$$\mu = e^{\int g(z) \mathrm{d}z}$$

   is an integrating factor for $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$. (**4**)

4. Consider the nth order ($n \geq 2$) homogeneous equation on an interval $I = [a, b]$:

$$L[y] = \sum_{i=0}^{n} p_i(x) y^{(n-i)} = 0,$$

where $y^{(k)}$ denotes the kth derivative of y and $p_0(x) \equiv 1$. Assume that the functions p_i's are continuous on I. Let the functions $y_1, \ldots, y_n$ satisfy $L[y_i] = 0$, $\forall i \in \{1, \ldots, n\}$. Define the Wronskian as

$$W(y_1, \ldots, y_n) := |A_{n,n}|,$$

where $A_{n,n}$ is the matrix defined as below :

$$A_{n,n} := \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

(a) Show that either W is identically 0 in I or $\min_{x \in I} |W(x)| > 0$ for $n = 3$. Generalize it to $n \geq 4$. (**8**)

(b) Assuming the above statement for $n \geq 2$, show that if $y_1, \ldots, y_n$ are linearly independent solutions then any function y_g satisfying $L[y_g] = 0$ can be expressed as a linear combination of $y_1, \ldots, y_n$. (**6**)

5. Find two linearly independent power series solutions about $x = 0$ for the differential equation

$$y^{''} - 2xy^{'} + \lambda y = 0$$

and explain for what values of λ is one of the solutions a polynomial. (**8**)

6. Consider the *Bessel's equation* for a real-number $p \geq 0$:

$$x^2 y^{''} + xy^{'} + (x^2 - p^2)y = 0.$$

(a) Let y_p be a non-trivial solution of Bessel's equation on the positive x-axis. If $0 \leq p \leq \frac{1}{2}$ then show that every interval of length π contains at least one zero of y_p. If $p > \frac{1}{2}$, then show that every interval of length π contains at most one zero of y_p. (**7**)

(b) Explain what is a Frobenius series solution and find the indicial equation of the Bessel's equation at $x = 0$. For what values of the roots, is the existence of a Frobenius series solution guaranteed ? (**5**)

(c) For $p = 1$, show that the equation can have only one non-trivial Frobenius series solution and describe the form of the single Frobenius series solution. (**6**)